A Hybrid Markov-Functional Model with Simultaneous Calibration to Interest Rate and FX Smile

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Pricing of Hybrid IR/FX products:

Hybrid IR/FX product = product that depends simultaneously on an (stochastic) interest rate and a (stochastic) FX rate.

Model requirements:

- Calibrates to (reproduces) market prices of European IR options for all maturities T_i and **all Strikes** K ($\hat{=}$ IR smile calibration)
- Calibrates to market prices of European FX options for all maturities T_i and **all Strikes** K ($\hat{=}$ FX smile calibration)
- Fast and Robust implementation

Proposed solution: IR/FX Markov Functional Model:

Main Features:

- Calibrates IR smile perfectly
- Calibrates FX smile perfectly
- 2D Lattice implementation (low dimensional and Markovian) $\Rightarrow$ Fast and Robust
- Additionally calibrates Co-terminal Swaptions

Universal Pricing Theorem (general result in derivative pricing):

Under some assumptions the value $V(t)$ of a financial product at time t is given by the conditional expectation with respect to a uniquely determined measure $\mathbb{N}$:

$$\frac{V(t)}{N(t)} = E^{\mathbb{N}}\left[\frac{V(s)}{N(s)}|\mathcal{F}_t\right] \qquad \forall s < t \quad ,$$

where N is some reference product (Numeraire).

$\mathbb{N}$ is such that for all financial products the above holds.

Markov-functional:

An interest rate model is said to be Markov-Functional if there exists some numeraire pair $(N, \mathbb{N})$ and some real-valued stochastic process X such that:

- The process X is a Markov process under $\mathbb{N}$.
- The pure discount bond prices are of the form $P(S; t, X_t(\omega))$, $0 \leq t \leq S$ for all $S \leq \hat{T}$.
- The numeraire is a price process of the form $N(t, X_t(\omega))$

$P(t,\cdot)$, $N(t,\cdot)$ are Functionals.

One possible specification:

- Numeraire: $N(t, X_t(\omega)) = P(\hat{T}; t, X_t(\omega))$
- $P(S; S, \cdot) \equiv 1$
- $dX_t = \sigma^x(t) dW_t^{(1)} \quad X_0 = x_0$

$$\Rightarrow P(S; t, X_t) = N(t, X_t) E_{\mathbb{N}}\left[\frac{P(S;S,X_S)}{N(S,X_S)}|\mathcal{F}_t\right]$$

The Hybrid Markov-Functional Model:

In addition to the single currency Markov-Functional interest rate model we introduce

- a process Y which is a Markov process under the measure $\mathbb{N}$
- a second Underlying as a price process, a stochastic process of the form

$$U(t, Y_t(\omega)), \quad 0 \leq t \leq \hat{T}$$

One possible specification:

- choose parameterized functional form of U
- $dY_t = \mu^y(t, X_t, Y_t)dt + \sigma^y(t) dW_t^{(2)}, \quad Y_0 = y_0$

such that: $\frac{U(t)}{N(t)} = E_{\mathbb{N}}[\frac{U(s)}{N(s)}|\mathcal{F}_t] \quad \forall s > t$ (no-arbitrage condition)[a]

[a] This can be satisfied by choosing the right drift of the driving process Y.

The Two-Factor Cross-Currency Model:

- stochastic domestic interest rates
- stochastic FX
- deterministic foreign interest rates

$U(T_i, Y_{T_i}) = FX(T_i, Y_{T_i})\tilde{P}(T_n; T_i)$ where $\tilde{P}$ denotes the deterministic price process of the foreign discount bond.

Here the no-arbitrage condition results in the following drift condition for Y:

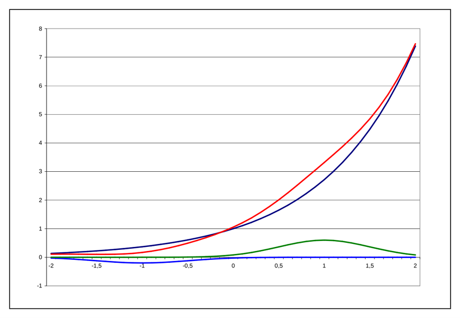
$$\frac{FX(T_i, Y_{T_i})\tilde{P}(T_{i+1}; T_i)}{N(T_i, X_{T_i})} = E_{\mathbb{N}}\left[\frac{FX(T_{i+1}, Y_{T_{i+1}})}{N(T_{i+1}, X_{T_{i+1}})}|\mathcal{F}_{T_i}\right] \quad \text{(drift condition)}$$

A FX Funktional to calibrate a general FX Vola Surface:

$$\eta \to FX(T_i, \eta)$$

$$\begin{aligned} FX(T_i, \eta) &= a(T_i) \cdot \exp(b(T_i) \cdot \eta) \\ &\quad + d_1(T_i) \cdot \exp(-c_1(T_i) \cdot (\eta - m_1(T_i))^2) \\ &\quad + d_2(T_i) \cdot \exp(-c_2(T_i) \cdot (\eta - m_2(T_i))^2) \end{aligned}$$

m = state of the correction
d = impact of the correction
c = radius of the correction ($c > 0$)



The calibration parameters of the model:

First dimension (IR): model entity	First dimension (IR): market entity	Second dimension (FX): model entity	Second dimension (FX): market entity
Numeraire functional	caplet smile	$b(t)$	ATM FX-option
		$d(t)$,$m(t)$	in/out-of-the-money FX-options
$\sigma^x(t)$	Swaption prices	$\sigma^y(t)$	Autocorrelation of FX-products

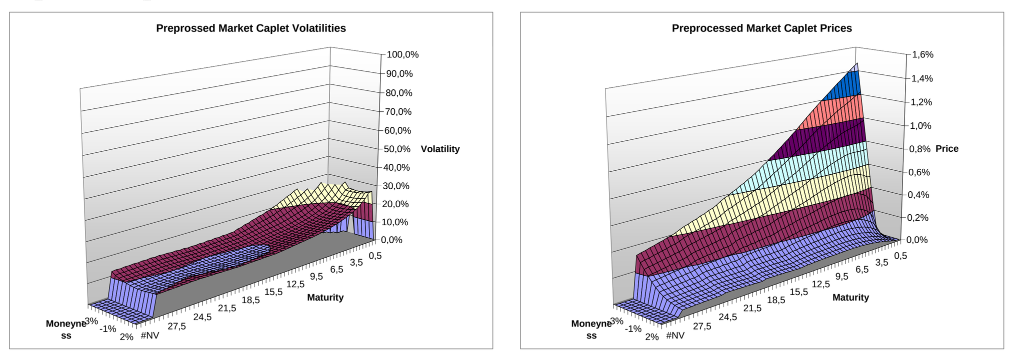
- $\mu^y(t, X_t, Y_t)$ has to solve the no-arbitrage / drift condition
- $c(t)$ stabilizes the calibration procedure

The General Calibration Procedure:

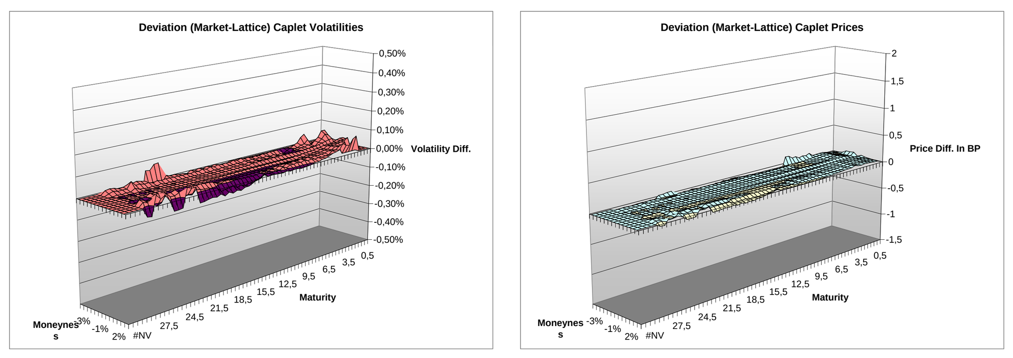
- for $T_0, \ldots, T_n$
  - optimize the FX functional with a multi-dimensional root finder by fitting model to market prices
    - * set FX parameters
    - * solve and set corresponding drift
    - * return the error between the market and model prices of specified FX options to the solver
    - * continue the optimization until a good smile fit
  - set FX to optimum
  - go to the next time step

Model output Caplets:

(i) Caplet sample market data:

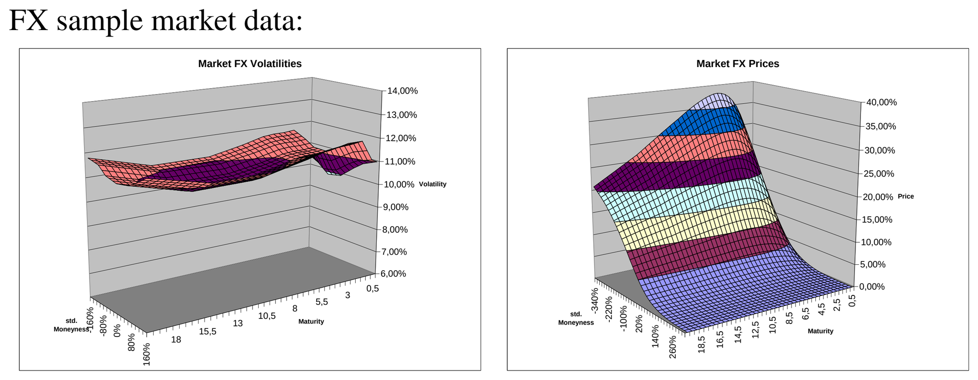


(ii) The deviation we observe in the model:

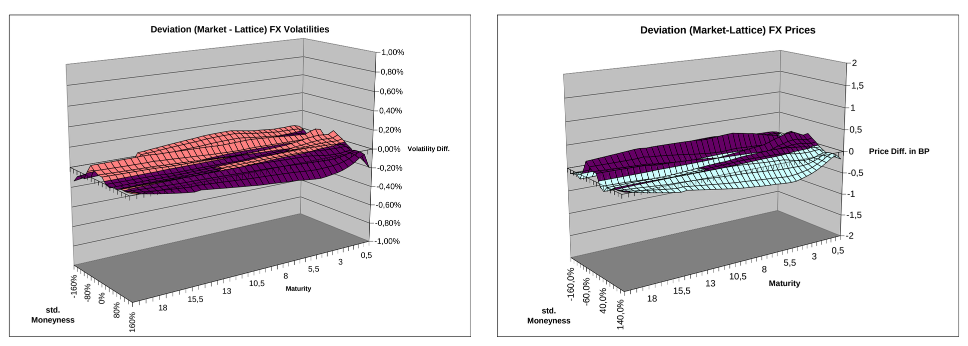


Model output FX-options:

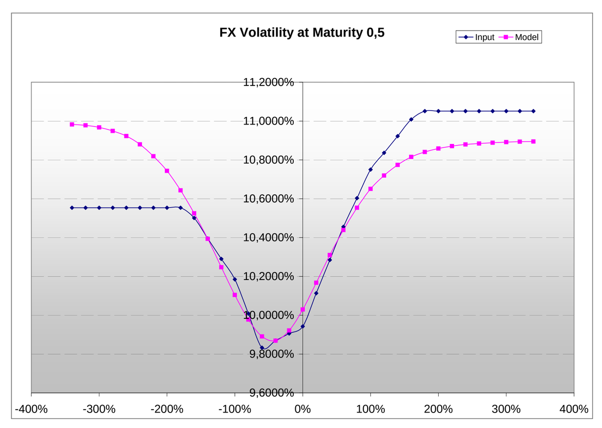
(i) FX sample market data:

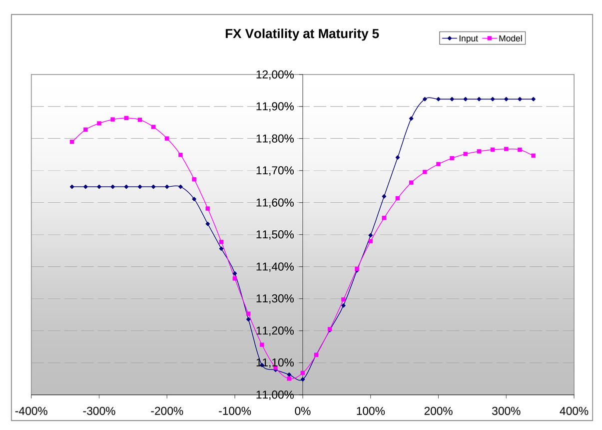


(ii) The deviation we observe in the model:



Market vs. Model FX-smile:





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